$\ddot{x} = ax$ First, Ansatz is $x(t) = A\cdot e^{\alpha t}$

$$\cancel{A}\alpha^2\cancel{e^{\alpha t}} = a\cancel{A}\cdot\cancel{e^{\alpha t}}$$

$$\alpha = \pm\sqrt{a} \Rightarrow x(t) = Ae^{\sqrt{a}t} + Be^{-\sqrt{a}t} \quad \text{(general solution)}$$

From here we have two choices.

1. if $a<0 \Rightarrow$ more physical system (SHO)

Then, we can define smth like $\omega^2 \equiv -a$

$$\alpha = \pm\sqrt{-\omega^2} = \boxed{\pm\omega i} \Rightarrow \boxed{x(t) = Ae^{i\omega t} + Be^{-i\omega t}}$$

Using $e^{i\theta} = \cos(\theta) + i\sin(\theta)$

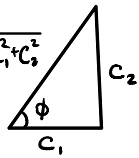
even odd functions

We get $A(\cos(\omega t) + i\sin\omega t) + B(\cos(\omega t) - i\sin(\omega t))$

$\boxed{x(t) = C_1\cos(\omega t) + C_2\sin(\omega t)} \Rightarrow$ fully "Real" equation.

From here, we can "dance" further:

$A \equiv \sqrt{C_1^2 + C_2^2}$



$$x(t) = A\cdot(\cos\phi\cos\omega t + \sin\phi\sin\omega t)$$

$$\boxed{x(t) = A\cdot\cos(\omega t - \phi)}$$

$$\cos\phi = \frac{C_1}{A}$$

2. if $a>0$ (less physical case) $\Rightarrow \alpha^2 = a \Rightarrow x(t) = A\cdot e^{+\alpha t} + Be^{-\alpha t}$.

↳ $\boxed{x(t) = C\cdot\cosh(\alpha t) + D\sinh(\alpha t) \;\&\; C_2\cosh(\alpha t + \phi)}$

For $\ddot{x} + 2\gamma\dot{x} + ax = 0$ Ansat: $x(t) = A\cdot e^{\alpha t}$

$$A\alpha^2 e^{\alpha t} + 2\gamma A\alpha e^{\alpha t} + aAe^{\alpha t} = 0 \Rightarrow \alpha^2 + 2\gamma\alpha + a = 0$$

$$D: 4\gamma^2 - 4\cdot a = 4(\gamma^2 - a) \Rightarrow \alpha_{1/2} = \frac{-2\gamma \pm \sqrt{4(\gamma^2-a)}}{2} = -\gamma \pm \sqrt{(\gamma^2-a)}$$

$$x(t) = A\cdot e^{(-\gamma + \sqrt{\gamma^2-a})t} + Be^{(-\gamma - \sqrt{\gamma^2-a})t} = e^{-\gamma t}\cdot\left(Ae^{\Omega t} + Be^{-\Omega t}\right)$$

where $\Omega^2 = \gamma^2 - a$

Simple harmonic Motion

$-Kx = m\ddot{x} \Rightarrow \ddot{x} + \frac{K}{m}x = 0$ we solved it already!

$x(t) = A\cos\omega t + B\sin\omega t$, where $\omega = \sqrt{\frac{K}{m}}$

$A = x_0$

$B = v_0/\omega$

Damped SHM

$m\ddot{x} + b\dot{x} + Kx = 0 \Rightarrow \ddot{x} + \frac{b}{m}\dot{x} + \frac{K}{m}x = 0$, $\quad 2\gamma \equiv \frac{b}{m} \quad \omega^2 \equiv \frac{K}{m}$

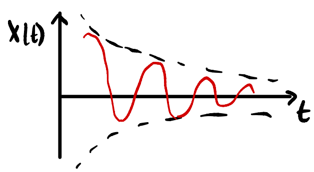
We know the solution: $x(t) = e^{-\gamma t}\left(Ae^{\Omega t} + Be^{-\Omega t}\right)$

with $\Omega \equiv \sqrt{\gamma^2 - \omega^2}$

Depending on γ ? ω (>, <, =) we have 3 cases.

Underdamping: $\gamma < \omega \Rightarrow$ Ω is imaginary.

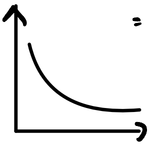
$$\tilde{\omega}^2 \equiv \omega^2 - \gamma^2 \Rightarrow \Omega = \tilde{\omega}i \Rightarrow x(t) = C\cdot e^{-\gamma t}\cdot\cos(\tilde{\omega}t + \phi)$$



Overdamping $\Rightarrow$ **Ω is real** $\Rightarrow \Omega \equiv \sqrt{\gamma^2-\omega^2}, \quad \gamma > \omega$

$x(t) = e^{-\gamma t}\left(Ae^{\Omega t} + Be^{-\Omega t}\right)$ → just goes to zero with large t.

$= A \cdot e^{-(\gamma-\Omega)t}$ $\qquad \gamma - \sqrt{\gamma^2-\omega^2} > 0$, so $x(t) \to 0$ for



This makes sense. $\gamma > \omega$ means damping wins.

Critical damping $\quad \Omega^2 = 0$. Our initial $x(t)$ solution is not valid! The real solution has $+\, t \cdot e^{-\gamma t}$

$$x(t) = e^{-\omega t}(C + t) \rightarrow \text{fastest to go to zero}$$

Compare it to Under/Overdamped cases for a fixed ω.