Driven Harmonic Oscillator.

Derivation no наизусть

We start from diff. equation! $m\ddot{x} + b\dot{x} + Kx = C_0 e^{i\omega_0 t}$

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{K}{m}x = C' e^{i\omega_0 t} \Rightarrow \ddot{x} + 2\gamma\dot{x} + \omega^2 x = C' e^{i\omega_0 t}$$

Ansat $x(t) = A \cdot e^{i\omega_0 t}$, plug in and see

$$A\omega_0^2 \cancel{e^{i\omega_0 t}} + 2\gamma i\omega_0 A_0 \cancel{e^{i\omega_0 t}} + \omega^2 \cdot A\cancel{e^{i\omega_0 t}} = C' \cancel{e^{i\omega_0 t}}$$

$$-\omega_0^2 A + 2\gamma i\omega_0 A + \omega^2 A = C'$$

$$\boxed{A = \frac{C'}{\omega^2 + 2\gamma i\omega_0 - \omega_0^2}} \checkmark \cdot e^{i\omega_0 t}$$

Now, our driving frequency is cosine function (just imagine)

$$m\ddot{x} + b\dot{x} + Kx = F\cdot\cos(\omega_0 t)$$

$$\ddot{x} + 2\gamma\dot{x} + \omega^2 x = \frac{F}{m}\cos(\omega_0 t) = \frac{F}{2m}\left(e^{i\omega_0 t} + e^{-i\omega_0 t}\right)$$

So our solution must be a sum of two functions!

preliminary solution

$$x(t) = \left(\frac{\frac{F}{2m}}{-\omega_0^2 + 2\gamma i\omega_0 + \omega^2}\right)e^{i\omega_0 t} + \left(\frac{\frac{F}{2m}}{-\omega_0^2 - 2\gamma i\omega_0 + \omega^2}\right)e^{-i\omega_0 t}$$

we have to eliminate i's

$$\frac{\left(\frac{F}{2m}\right)(\omega^2 - \omega_0^2 - 2\gamma\omega_0 i)}{(\omega^2 - \omega_0^2 + 2\gamma\omega_0 i)(\omega^2 - \omega_0^2 - 2\gamma\omega_0 i)} = \frac{\frac{F}{2m}(\overbrace{\omega^2 - \omega_0^2}^{a} - \overbrace{2\gamma\omega_0 i}^{b})}{\left((\omega^2 - \omega_0^2)^2 + 4\gamma^2\omega_0^2\right)}\left(\cos(\omega_0 t) + i\sin(\omega_0 t)\right)$$

$(a+bi)(a-bi)$

$\underbrace{a^2 + b^2}$

$$+ \frac{\frac{F}{2m}(\overbrace{\omega^2 - \omega_0^2}^{a} + \overbrace{2\gamma\omega_0 i}^{b})}{\left((\omega^2 - \omega_0^2)^2 + 4\gamma^2\omega_0^2\right)}\left(\cos(\omega_0 t) - i\sin(\omega_0 t)\right)$$

$$\frac{\frac{F}{2m}\left(a\cos\omega_0 t - \cancel{b\cos\omega_0 t} + \cancel{ai\sin\omega_0 t} - bi\sin\omega_0 t + a\cos\omega_0 t + \cancel{b\cos\omega_0 t} - \cancel{ai\sin\omega_0 t} - bi\sin\omega_0 t\right)}{\left((\omega^2 - \omega_0^2)^2 + 4\gamma^2\omega_0^2\right)}$$

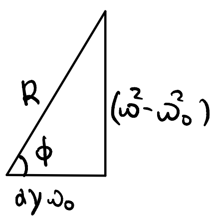
$$x(t) = \frac{\frac{F}{m}\left((\omega^2 - \omega_0^2)\cos\omega_0 t + 2\gamma\omega_0\sin\omega_0 t\right)}{\left((\omega^2 - \omega_0^2)^2 + 4\gamma^2\omega_0^2\right)}$$

Actually, a really smart move.

$$\boxed{R^2 \equiv (\omega^2 - \omega_0^2) + (2\gamma\omega_0)^2}$$

Now, we do some „tricks" define bottom to be R^2.

Then, $\frac{F}{mR}\left(\frac{(\omega^2 - \omega_0^2)}{R}\cos\omega_0 t + \frac{2\gamma\omega_0}{R}\sin\omega_0 t\right)$



Теперь, мы подгоним под формулу $\rightarrow \cos(a-b)$.

$$x(t) = \frac{F}{mR}\left(\cos(\omega_0 t - \phi)\right) \quad \text{where } \phi = \cos^{-1}\left(\frac{\omega^2 - \omega_0^2}{R}\right)$$

b/w $0 \le \phi \le \pi$, because sin must be positive

We also add our damped solution. Remember, linearity!

$$x(t) = \frac{F}{mR}\left(\cos(\omega_0 t - \phi)\right) + e^{-\gamma t}\left(Ae^{\Omega t} + Be^{-\Omega t}\right)$$

Resonance $\Rightarrow \omega_0 = \omega$